A Theory of Locally Convex Hopf Algebras
Part I. Basic Theory and Examples

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Motivation I: dissatisfaction

The **operator algebraic approach** (Woronowicz, Van Daele, Baaj-Skandalis, Kustermans-Vaes...) is already quite successful.

Question

Why another approach to topological quantum groups?

Personal dissatisfaction with the operator algebraic approach

1. **Technical complication**: modular theory, multiplier algebras, unbounded operators, manageable multiplicative unitaries...
2. **Limitation to the locally compact case**: description of some nice topological (quantum) groups that are *not locally compact*
3. **Seemingly unnatural axioms**: further and further away from Hopf algebras
4. **Inaccessibility to nice "functions"**: e.g. smooth functions
5. **The hopeless Haar measure/weight problem**

Motivation II: toolbox choice

The theory of operator algebras (OA) is very active, while the theory of locally convex spaces (LCS) seems almost dead by comparison.

Question

Two powerful and mature toolbox: OA vs LCS, why the latter?

Answer

- LCS has by far the most **systematic duality theory** in functional analysis.
- LCS is **more flexible**: e.g. smooth functions, distributions...
- Categorically speaking, LCS enjoys much **nicer formal properties**, e.g. better universal properties for tensor products.
- As for describing topological spaces, LCS can **go beyond the locally compact setting** (next talk).

The main idea and subtleties

- A **locally convex Hopf algebra** should be a *complete* LCS H equipped with the usual structure maps $(m, \Delta, \eta, \varepsilon, S)$, but with the algebraic tensor products replaced by some *suitable completed topological tensor products.*
- If $H' \otimes H' = (H \otimes H)'$ etc., then transposing the above structure maps yields H' as a locally convex Hopf algebra dual to H.

Main subtleties

- Many possible choices for the topologies on H', as well as for the type of tensor products in $H \otimes H$ and $H' \otimes H'$ etc.
- The dualities of the type $H' \otimes H' = (H \otimes H)'$ often fails.
- A good theory **should contain at least a large supply of interesting examples** of classical and quantum topological groups. So we **can not impose too strict restriction**.

Peeking ahead

We will now describe how to realize the above main idea and overcome the aforementioned difficulties. In particular, we shall cover (non-locally compact aspects will be in the next talk):

- how to formulate a good notion of locally convex Hopf algebras, as well as the corresponding duality;
- how a new type of duality (resp. reflexivity) in addition to the strong one, termed **polar reflexivity** (resp. **polar reflexivity**) come into play and give examples for this new duality phenomenon;
- how to describe arbitrary Lie groups in this framework using only smooth functions;
- how to resolve the duality problem for real and complex Hopf algebras in this framework;
- how to include compact/discrete quantum groups into this theory and give their characterization.

Locally convex direct sums and products

Locally convex direct sums

Given a family $(E_i)_{i\in I}$ of LCS, set $E := \oplus_{i\in I} E_i$.

- There is a unique finest locally convex topology τ on E making each $E_i \hookrightarrow E$ continuous, called the **locally convex direct sum topology**.
- If each E_i is complete, then so is (E, τ).
- A neighborhood basis at 0 in E consists of sets of the form $\Gamma(\cup_i V_i)$ (absolutely convex hull), where $V_i \in \mathcal{N}_{E_i}(0)$.
- Some subtlety: in $\oplus_{n\geq 1}\mathbb{R}$, the sequence $(1/n)e_n$ does *not* converge to 0 even $1/n \longrightarrow 0$ (take $V = \Gamma(\cup_n V_n)$), where $V_n =]-1/n, 1/n[$ in the n-th copy of $\mathbb{R}$.

Locally convex direct products is more familiar and much easier—just take the product topology.

Some examples of direct sums and products

Below the scalar field is $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

- (Hopf algebras and their strong duals) Given any vector space V, choose a basis (axiom of choice), $V = \oplus_{i \in I} \mathbb{K}$. The corresponding locally convex direct sum topology on V is **the finest locally convex topology** on V. Such V is (strongly) reflexive, with the strong dual being canonically isomorphic to $\prod_{i \in I} \mathbb{K}$ (The only weakly complete spaces up to isomorphism).
- (Lie groups and their strong duals) Given an arbitrary (real) Lie group G, let G_0 be its neutral component. Equipped with the **topology of compact convergence on all derivatives**, we have

$$C^\infty(G) = \prod_{X \in G/G_0} C^\infty(X) \simeq C(G_0)^{[G:G_0]}.$$

Such $C^\infty(G)$ is again strongly reflexive, with strong dual being isomorphic to the locally direct sum $\oplus_{i \in [G:G_0]} C(G_0)'_b$.

Locally convex topologies determined by dualities

- A **duality pairing** $\langle E, F \rangle$ is a non-degenerate bilinear map $E \times F \to \mathbb{K}$.
- If E is a LCS, the topology on E is **compatible with the pairing** if $F \to E'$, $y \in F \mapsto \langle \cdot, y \rangle$ is well-defined bijection.
- There exists a coarsest locally convex topology on E that is compatible with the pairing $\langle E, F \rangle$, called the **weak topology** and is denoted by $\sigma(E, F)$.
- There exists a finest locally convex topology on E that is compatible with the pairing $\langle E, F \rangle$, called the **Mackey topology** and is denoted by $\tau(E, F)$.
- A LCS E is called a **weak** (resp. **Mackey**) **space** if its topology is the weak (resp. Mackey) topology $\sigma(E, E')$ (resp. $\tau(E, E')$) for the canonical pairing $\langle E, E' \rangle$.
- Weak and Mackey topologies behave well with the dualities between locally convex direct sums and products.

Topology of $\mathfrak{S}$-convergence

- $\mathfrak{S}$ a class of subsets of a fixed LCS E.
- S_F the collection of continuous seminorms on a LCS F.
- $\mathcal{L}(E, F)$ the space of continuous linear operators.
- For $A \in \mathfrak{S}$, $q \in S_F$, $\varphi \in \mathcal{L}(E, F)$ define

  $$p_{A,q}(\varphi) = \sup\left\{q\big(\varphi(x)\big) \,\middle|\, x \in A\right\},$$

  then $p_{A,q}$ is a seminorm on $\mathcal{L}(E, F)$.
- The **topology of $\mathfrak{S}$-convergence**, or the **$\mathfrak{S}$-topology** on $\mathcal{L}(E, F)$ is the linear topology generated by all seminorms of the form $p_{A,q}$, $A \in \mathfrak{S}$, $q \in S_F$. We thus get a LCS $\mathcal{L}_{\mathfrak{S}}(E, F)$.
- E.g. $\mathcal{L}_b(E, F)$–**bounded convergence**; $\mathcal{L}_c(E, F)$-**(pre)compact convergence**, $\mathcal{L}_s(E, F)$-**simple convergence**.
- When $F = \mathbb{K}$, the above becomes E'_b, E'_c and E'_s.

Bornological spaces and completeness on the dual

- E a LCS, $B \subseteq E$ is called **bornivorous** if it absorbs all bounded sets, i.e. for all bounded $A \subseteq E$, we have $A \subseteq tB$ for all large $t \in \mathbb{K}$.
- E is called **bornological** if every absolutely convex bornivorous set of E is a neighborhood of 0.
- E.g. all metrizable LCS, inductive limit (next talk, in particular direct sums) and countable direct products (unknown without countability) of bornological spaces.

Here's an important result on completeness.

Proposition (Bourbaki)

If E is bornological, F complete, $\mathfrak{S}$ contains the images of all null sequences in E, then $\mathcal{L}_{\mathfrak{S}}(E, F)$ is complete.

More “nice” spaces

Let E be a LCS, we say E is

- a **Fréchet space** or an F-space, if it is complete and metrizable, $(\mathcal{F})$ denotes the class of all F-spaces, and $(\mathcal{F}_c')$ the class of their polar duals;
- **barrelled**, if each barrel (non-empty closed absolutely convex set) is neighborhood of 0;
- **Montel**, or an M-space, if it is barrelled and each bounded set is relatively compact, $(\mathcal{M})$ denotes the class of all Montel spaces.

Examples:

- Fréchet: $C(G)$ (G locally compact) and $C^\infty(G)$ (G a Lie group) for second countable G.
- Montel: $C^\infty(G)$ for arbitrary Lie group G, any V with the finest locally convex topology.
- Barrelled: Baire spaces (in particular, F-spaces), inductive limits (next talk, in particular direct sums) of barrelled spaces.

Polar and Strong Reflexivity

Definition

A duality pairing $\langle E, F \rangle$ is called **polar (resp. strongly) reflexive**, if E, F can be identified as the polar (resp. strong) dual of each other via this pairing. We say a locally convex space E is polar (resp. strongly) reflexive, if the canonical pairing $\langle E, E' \rangle$ is so.

- (Brauner 1973) All F-spaces are polar reflexive.
- The polar and strong duals of an F-space are always complete.
- All M-spaces are strongly reflexive.
- Let G be a discrete group. Unless G is finite, $\ell^1(G)$ is never strongly reflexive; by contrast, it is always polar reflexive.
- We will later make $\ell^1(G)$ into a locally convex Hopf algebra that is polar reflexive but *not* strongly reflexive.
- Polar dual is also termed **stereotype dual** (Akbarov).

Compatible topologies on the tensor product

Let E be a LCS. Recall that $A \subseteq E'$ is called **equicontinuous** if $A \subseteq V^\circ$, i.e. $|\langle a, v \rangle| \le 1$ for all $a \in A$, $v \in V$, for some $V \in \mathcal{N}_E(0)$.

Definition (Grothendieck)

Let E, F be lcs, then a locally convex topology τ on $E \otimes_{\text{alg}} F$ (algebraic tensor product) is called **compatible**, if

1. the canonical bilinear map $E \times F \longrightarrow E \otimes_\tau F$ is **separatedly continuous**;
2. $u' \otimes v' \in (E \otimes_{\text{alg}} F)'$ for all $u' \in E'$, $v' \in F'$;
3. $A \otimes B := \{f \otimes g \mid f \in E', g \in F'\}$ is **equicontinuous** on $E \otimes_\tau F$ for all equicontinuous $A \subseteq E'$ and $B \subseteq F'$.

Notation

The completion of $E \otimes_\tau F$ will be denoted by $E \overline{\otimes}_\tau F$.

Topological tensor products

Let $E, F \in$ LCS and $\chi : E \times F \longrightarrow E \otimes_{\text{alg}} F$ canonical.

- The injective (resp. inductive) tensor product topology is the **coarsest** (resp. **finest**) **compatible topology** on $E \otimes_{\text{alg}} F$, the resulting LCS is denoted by $E \otimes_{\varepsilon} F$ (resp. $E \otimes_{\iota} F$), called the **injective** (resp. **inductive**) **tensor product**.
- $\chi : E \times F \longrightarrow E \otimes_{\iota} F$ is the universal **separately continuous** bilinear map from $E \times F$.
- There is also a unique compatible topology $\mathfrak{T}_{\pi}$ such that $\chi : E \times F \longrightarrow E \otimes_{\pi} F := (E \otimes_{\text{alg}} F, \mathfrak{T}_{\pi})$ is the universal **jointly continuous** bilinear map from $E \times F$. We call $E \otimes_{\pi} F$ the **projective tensor product**.
- $\otimes_{\pi}$ and $\otimes_{\varepsilon}$ are used much more often than $\otimes_{\iota}$, partly because $E \otimes_{\iota} F = E \otimes_{\pi} F$ in many cases (but not always).

Nuclear spaces and the approximation property

- We have a comparison map $E \otimes_\pi F \longrightarrow E \otimes_\varepsilon F$, for the injective tensor product $E \otimes_\varepsilon F$ is the coarsest compatible topology.
- We call E **nuclear** if this comparison map is an isomorphism for any F.
- We say E has **the approximation property**, or (AP), if finite rank operators are dense in $\mathcal{L}_c(E)$.
- Notation: $(\mathcal{N})$-the class of nuclear spaces, $(\mathcal{AP})$-the class of spaces having (AP).
- $(\mathcal{AP})$ is stable under taking $\overline{\otimes}_\pi$ (equivalently $\overline{\otimes}_\varepsilon$), arbitrary direct products and sums.
- $(\mathcal{N})$ is stable under taking $\overline{\otimes}_\varepsilon$, arbitrary direct products and *countable* direct sums.

More on nuclear spaces and (AP)

- $(\mathcal{N}) \subseteq (\mathcal{AP})$, i.e. nuclear implies (AP).
- Complete nuclear spaces are Montel.
- For any second countable smooth manifold M, we have $C^\infty(M) \in (\mathcal{N})$ (in particular for finite dimensional spaces).
- For any Lie group G, we have $C^\infty(G) \in (\mathcal{N})$.
- $\oplus_{i \in I} \mathbb{K} \in (\mathcal{N})$ if and only if I is at most countable.
- For $E \in (\mathcal{F})$, we have $E \in (\mathcal{N})$ if and only $E'_b \in (\mathcal{N})$.
- For any locally compact X, $C(X) \in (\mathcal{AP})$ (this will be generalized in the next talk).
- For any Radon measure μ on X, $L^p(X, \mu) \in (\mathcal{AP})$, $1 \le p \le \infty$.
- For $E \in (\mathcal{F})$, we have $E \in (\mathcal{AP})$ if and only $E'_c \in (\mathcal{AP})$.
- After Grothendieck work, it is an open problem for quite some time to determine whether there every LCS is in $(\mathcal{AP})$, which is finally answered in the negative by Enflo in 1973.

The Buchwalter duality

Theorem (Buchwalter, 1972)

Let E, F be F-spaces, with one of them having(AP), then ($\overline{\otimes}_\pi$ and $\overline{\otimes}_\varepsilon$ are ***polar duals*** *of each other)*

- $(E\overline{\otimes}_\pi F)'_c = E'_c\overline{\otimes}_\varepsilon F'_c$ *and* $(E'_c\overline{\otimes}_\varepsilon F'_c)'_c = E\overline{\otimes}_\pi F$;
- $(E\overline{\otimes}_\varepsilon F)'_c = E'_c\overline{\otimes}_\pi F'_c$ *and* $(E'_c\overline{\otimes}_\pi F'_c)'_c = E\overline{\otimes}_\varepsilon F$.

- $(\mathcal{F})$ is stable under $\overline{\otimes}_\pi$ and $\overline{\otimes}_\varepsilon$.
- $E'_b = E'_c$ if the F-space E is also Montel (so precompactness = boundedness). We set $(\mathcal{FM}) = (\mathcal{F}) \cap (\mathcal{M})$.
- $(\mathcal{FM})$ is stable under $\overline{\otimes}_\varepsilon$.
  Open question: stability of $(\mathcal{FM})$ under $\overline{\otimes}_\pi$?
- $(\mathcal{F}) \cap (\mathcal{AP})$ is stable under $\overline{\otimes}_\varepsilon$; and $E \in \mathcal{F}$ has (AP) if and only if E'_c does.
- One can drop (AP) by using ε-**product** of Schwartz.

Locally Convex Hopf Algebras

Definition

A complete locally convex space H is called a **projective Hopf algebra**, or a π-Hopf algebra, if it is equipped with a multiplication m, a comultiplication Δ, a unit η, a counit ε and an antipode S, satisfying all axioms of a Hopf algebra with the algebraic tensor product replaced by $\overline{\otimes}_\pi$. An **injective**, or ε-**Hopf algebra**, as well as an **inductive**, or ι-**Hopf algebra**, are defined similarly.

The axioms for the structure maps are (note the symmetry)

$$\begin{gathered}
m(m \otimes \mathrm{Id}) = (\mathrm{Id} \otimes m) m, \qquad m(\eta \otimes \mathrm{Id}) = \mathrm{Id} = m(\mathrm{Id} \otimes \eta), \\
(\Delta \otimes \mathrm{Id})\Delta = (\mathrm{Id} \otimes \Delta)\Delta, \qquad (\mathrm{Id} \otimes \varepsilon)\Delta = \mathrm{Id} = (\varepsilon \otimes \mathrm{Id})\Delta, \\
\varepsilon m = \varepsilon \otimes \varepsilon, \qquad \Delta\eta = \eta \otimes \eta \\
(m \otimes m)(\mathrm{Id} \otimes \sigma \otimes \mathrm{Id})(\Delta \otimes \Delta) = \Delta m \ (\sigma \text{ being the flip}), \\
m(S \otimes \mathrm{Id})\Delta = \eta \circ \varepsilon = m(\mathrm{Id} \otimes S)\Delta.
\end{gathered} \tag{1}$$

Reflexive Locally Convex Hopf Algebras

Definition

Let $\tau, \sigma \in \{\varepsilon, \pi, \iota\}$. We say a τ-Hopf algebra H is (τ, σ)-**polar reflexive**, if we have canonical pairings $\left\langle H^{\overline{\otimes}_\tau k}, H_c'^{\overline{\otimes}_\sigma k} \right\rangle$ for $k = 1, 2, 3, 4$, and they are all polar reflexive.
Reflexivity is defined similarly by replacing the polar duals with strong duals.

- The (strong) dual $H' = H_b'$ of a (τ, σ)-polar reflexive Hopf algebra H has a canonical σ-Hopf algebra structure by taking transposes of the structure maps, called the **dual of** H. Similarly, for the polar duals.
- Four fold tensor products are needed since the axiom that comultiplication is multiplicative involves four fold tensor products.

Theorem on Polar Reflexivity

The Buchwalter duality immediately yields the following.

Theorem (W, 24)

- *If H is an ε-Hopf algebra (resp. π-Hopf algebra) of class $(\mathcal{F}) \cap (\mathcal{AP})$, then H is (ε, π)-polar reflexive (resp. (π, ε)-reflexive), and the polar dual H'_c is of class $(\mathcal{F}'_c)$.*
- *If H is an ε-Hopf algebra (resp. π-Hopf algebra) of class $(\mathcal{F}'_c) \cap (\mathcal{AP})$, then H is (ε, π)-polar reflexive (resp. (π, ε)-reflexive), and the polar dual H'_c is of class $(\mathcal{F})$.*

E.g. if G is a σ-compact locally compact group, then $C(G)\overline{\otimes}_\varepsilon C(G) = C(G \times G)$, and $C(G) \in (\mathcal{F}) \cap (\mathcal{AP})$ has a canonical ε-Hopf algebra structure induced by the group operations on G (more general result in next talk).

Another Polar Reflexive Example

Let Γ be an arbitrary discrete group. Then $\ell^1(\Gamma \times \Gamma) = \ell^1(\Gamma)\overline{\otimes}_\pi \ell^1(\Gamma)$. The group Hopf algebra $\mathbb{C}[\Gamma]$ extends in a unique way to a **π-Hopf algebra structure** on $H = \ell^1(\Gamma)$.

- H is (π, ε)-polar reflexive since $\ell^1(\Gamma)$ is a Banach space.
- All L^p-spaces ($(p \in [0, +\infty])$) with respect to a Radon measure on a locally compact space has (AP), in particular, $\ell^1(\Gamma)$.
- The polar dual H'_c has $\ell^\infty(\Gamma)$ as the underlying space, but the topology is the one of compact convergence with respect to the duality with $\ell^1(\Gamma)$, which is no longer normable. It is interesting to note that H'_c has an ε-Hopf algebra structure!
- When Γ is infinite, H **can not be** (π, τ)-**reflexive** for any compatible τ, in particular for $\tau \in \{\pi, \varepsilon, \iota\}$.
- One can still recover Γ as the group of group-like elements of H. So we obtain a duality result for arbitrary discrete groups that is not accessible using only (strong) reflexivity.

Projective vs inductive tensor product

- We have $E\overline{\otimes}_\pi F = E\overline{\otimes}_\iota F$ if $E, F \in (\mathcal{F})$ or E, F are both strong duals of F-spaces (separately continuous bilinear maps from $E \times F$ are continuous).
- If $(E_i)_{i\in I}$, $(F_j)_{j\in \mathcal{J}}$ are families of spaces in $(\mathcal{FM}) \cap (\mathcal{AP})$, then

$$\left((\prod_i E_i)\overline{\otimes}_\varepsilon(\prod_j F_j)\right)'_b = \left(\prod_{i,j} E_i\overline{\otimes}_\varepsilon F_j\right)'_b$$
$$= \bigoplus_{i,j}(E_i\overline{\otimes}_\varepsilon F_j)'_b = \bigoplus_{i,j}(E_i)'_b\overline{\otimes}_\pi(F_j)'_b$$
$$= \bigoplus_{i,j}(E_i)'_b\overline{\otimes}_\iota(F_j)'_b = \left(\bigoplus_i (E_i)'_b\right)\overline{\otimes}_\iota\left(\bigoplus_j (F_j)'_b\right).$$

- $\overline{\otimes}_\varepsilon$ (resp. $\overline{\otimes}_\iota$) commutes with direct products (resp. sums), and the involved spaces are Mackey.

Theorem on Strong Reflexivity

Theorem (W,24)

- *Let H be an ε-Hopf algebra. If as a locally convex space, H is isomorphic to a product of spaces in $(\mathcal{FM}) \cap (\mathcal{AP})$, then as an ε-Hopf algebra, it is (ε, ι)-reflexive. In particular, this applies when the locally convex space H is isomorphic to a product of spaces in $(\mathcal{FN})$.*
- *Dually, if H is an ι-Hopf algebra with H decomposes as the direct sum of strong duals of spaces in $(\mathcal{FM}) \cap (\mathcal{AP})$, then H is (ι, ε)-reflexive.*

- The proof follows from the computation in the previous slide.
- The special case where H is a nuclear F-space (or the strong dual of such spaces) is already studied by Bonneau, Flato, Gernstenhaber & Pinczon (BFGP, 1994) under the term "well-behaved topological Hopf algebra".

Lie Groups

Let G be a (real) Lie group, second countable or not.

- The connected component G_0 of the neutral element is second countable.
- Equipped with the topology of compact convergence of all derivatives, $C^\infty(G)$ is a complete nuclear space, isomorphic to a $[G : G_0]$ copy of the nuclear F-space (hence FM) $C^\infty(G_0)$.
- $\mathscr{H} = C^\infty(G)$ has a ε(or π by nuclearity)-**Hopf algebra** structure induced by group operations.
- $\mathscr{H}$ is (ε, ι)-reflexive.
- One may recover G from $\mathscr{H}$ as a topological group (next talk). Since the compatible smooth structure on G is unique, we can essentially recover G as a Lie group from $\mathscr{H}$.

Duality of Hopf algebras–I. The problem

The duality problem for Hopf algebras

Let H be a Hopf algebra over k, and H' the linear dual. When $\dim_k H = \infty$, one **can not get a Hopf algebra structure on the dual** H' by transposing the corresponding structure maps for H.

The restricted dual (Sweedler?) as a partial workaround

Set $H^\circ := \{\omega \in H' \mid m^\intercal(\omega) = \omega m \in H' \otimes H' \subseteq (H \otimes H)'\}$, then H° becomes a well-defined Hopf algebra by transposing the corresponding structure maps of H.

But the duality $\langle H, H^\circ \rangle$ **can be degenerate**. Let $\Gamma = \langle a_k \mid a_{k+1} a_k = a_k a_{k+1}^2, k \in \mathbb{Z}/4\mathbb{Z} \rangle$ (**the Higman group**). Then $k[\Gamma]^\circ = k$, so all information about Γ is completely lost!

Duality of Hopf Algebras–II. Preparations

Some facts

- One can form arbitrary direct sums in the category of locally convex spaces, which preserves completeness.
- Writing any vector space V as a direct sum of one-dimensional spaces shows that there is a unique finest locally convex topology on V.
- Complete locally convex spaces whose topology coincides with the weak topology is isomorphic to a product of one dimensional spaces.
- Spaces of the above two types are reflexive, and are the strong duals of each other.
- Uncountable direct sums of one dimensional spaces is always **non-nuclear** (but is still Montel).

Duality of Hopf Algebras–III. Results and Comments

Theorem (W, 24)

Any classical Hopf algebra H can be seen as an ι-Hopf algebra equipped with the finest locally convex topology on H. It is then (ι, ε)-reflexive, with its strong dual being of class $(\mathcal{N})$.

- This solves completely the duality problem of Hopf algebras when the scalar field is $\mathbb{C}$ or $\mathbb{R}$.
- H is nuclear if and only if it is of countable dimension, in which case the result is already in (BFGP 1994).
- (W, 24) Classical Hopf algebras are precisely ι-Hopf algebras with the finest locally convex topology.
- (W, 24) The strong duals of classical Hopf algebras are precisely those ε-Hopf algebras (or π-Hopf algebras by nuclearity) that are complete weak spaces, or equivalently, isomorphic to a direct product of 1-dimensional spaces.

Compact and Discrete Quantum Groups

- When $\mathbb{K} = \mathbb{C}$, one can introduce the $*$-structure.
- Instead of taking transpose, the duality involving involution is given by $\langle x, \omega^* \rangle = \overline{\langle (Sx)^*, \omega \rangle}$, where $x \in H$, $\omega \in H'$.
- (Woronowicz) A compact quantum group $\mathbb{G}$ (Woronowicz) is completely characterized by $\mathrm{Pol}(\mathbb{G})$, which is a Hopf-$*$ algebra.
- Van Daele's version of $\widehat{\mathbb{G}}$ can now be described **without using multipliers**: $M(c_c(\widehat{\mathbb{G}}))$ is simply the strong dual $\mathrm{Pol}(\widehat{\mathbb{G}})'$ and $M(c_c(\widehat{\mathbb{G}}) \otimes c_c(\widehat{\mathbb{G}}))$ simply $\mathrm{Pol}(\widehat{\mathbb{G}})' \overline{\otimes}_\pi \mathrm{Pol}(\widehat{\mathbb{G}})'$ and is nuclear.
- (W, 24) One can now characterize discrete quantum groups as locally convex $*$-Hopf algebras that coincides with its weak topology, and is isomorphic as a product of matrix algebras as $*$-algebras.
- (W, 24) One can also characterize CQGs as locally convex $*$-Hopf algebras with the finest locally convex topology and a positive invariant integral.

This is the end of talk I.
Next time we will focus on the non-locally aspects of the theory.

Thank you!